$\mathbb{Z}/_3 \times \mathbb{Z}/_3$ acts non-trivial on $\mathbb{Z}/_7$

or

$\mathbb{Z}/_9$ " " $\mathbb{Z}/_7$

G acts on H as a group:

$G \times H \to H$

$(g,h) \mapsto h^g$ $\quad (g \cdot h)$

- $(h^{g_2})^{g_1} = h^{(g_1 g_2)}$ $\quad (g_1(g_2 \cdot h) = (g_1 g_2) h$
- $h^{e_G} = h$ $\quad (e_G \cdot h = h$
- $(h_1 \cdot h_2)^g = (h_1^g) \cdot (h_2^g)$ $\quad \cancel{\quad}$

Or

$\exists$ homo. $\varphi : G \to \mathrm{Aut}(H)$

$\Rightarrow$ $H \rtimes_\varphi G$

Fact: If φ is non-trivial

$H \rtimes_\varphi G$ non-abelian.

$$H \rtimes_\rho G \overset{\text{set}}{=} H \times G$$

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2^{g_1}, g_1 g_2)$$

$$\boxed{h_2^{g_1} := \rho(g_1)(h_2)}$$

ρ non-triv $\Rightarrow$ $\exists\, g, h$ s.t. $h^g \neq h$

$$(e_H, g) \cdot (h, e_G) = (e_H \cdot h^g, g \cdot e_G) = (h^g, g)$$

$$(h, e_G) \cdot (e_H, g) = (h, g) \quad \neq$$

$C_7 = \langle x \mid x^7 = e \rangle$

~~$\rho : C_3 \times C_3 \to \mathrm{Aut}(C_7) \cong \mathbb{Z}_6$~~

Non-triv

~~$\rho : C_9 \to \mathrm{Aut}(C_7) \cong \mathbb{Z}_6$~~

Fact 2

$$C_3 \times C_3 = \langle a, b \mid a^3 = 1,\ b^3 = 1,\ ab = ba \rangle$$

$$\downarrow$$

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2 (b)

$$C_7 \rtimes_f C_9 \xleftarrow{\text{UMP}} \langle \tilde{x}, \tilde{y} \mid \tilde{x}^7 = 1,\ \tilde{y}^9 = 1,\ \tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{x}^2 \rangle = L$$

$\tilde{x} \mapsto (x, 1)$

$\tilde{y} \mapsto (1, y)$

$$\boxed{f(y)(x) = x^2}$$

$$\tilde{y}\tilde{x} = \tilde{x}^2 \cdot \tilde{y}$$

- Map exists by UMP
- Onto : not hard
- ETS: $\#L \leq 63$ ✓

Fact 2 $\quad Aut(\mathbb{Z}/p) \xleftarrow{\cong} (\mathbb{Z}/p)^{\times} = (\mathbb{Z}/p \setminus \{0\}, \cdot)$

$(m \mapsto n \cdot m) \leftarrow n$

Fact 3 $\quad (\mathbb{Z}/p)^{\times}$ is cyclic

$(\mathbb{Z}/7)^{\times} = \cancel{\langle 2 \rangle}$ $\quad 2 \mapsto 2^2 \mapsto 2^3 = 8$

$= \langle 3 \rangle$

$3 \mapsto 3^2 \mapsto 18$

$2 \mapsto 6 \mapsto 18 = 4 \mapsto 10$

$4 \mapsto 12 = 5 \mapsto 15 = 1$

$\rho : C_3 \times C_3 \to Aut(C_7) \cong \mathbb{Z}/6$

$\cong C_9$

$C_7 = \langle x \mid x^7 = 1 \rangle$

$Aut(C_7) \ni \alpha \quad \alpha(x) = x^3$

$\langle \alpha \rangle = Aut(C_7)$

$C_9 = \langle y \mid y^9 = 1 \rangle$

$C_7 = \langle x \mid x^7 = 1 \rangle$

$\rho(y) = \alpha \in Aut(C_7) \quad \alpha(x) = x^2$

$|\alpha| = 3$